Super-Resolution in several dimensions and some applications

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The Problem

- Consider the discrete positive measure

$$\mu := \sum_{i=1}^{k} c_i \delta_{x_i}, c_i > 0$$

$x_j \in X \subseteq \mathbb{R}^n$.

- "Measurements" of the measure can be thought as

$$y_j := \int \Phi_j d\mu, \quad j = 1, ..., m,$$

for some functions Φ_j, $j = 1, \ldots, m$ which are known or can be estimated.

The goal of super-resolution process is to recover μ knowing ONLY y_j and Φ_j, $j = 1, \ldots, m$.

Applications

Cubature Formulas

- In $\mathbb{R}$, a cubature formula is a way to approximate the integral $\int_a^b f(t)\mathrm{d}t$.
- The integral is approximated as

$$\int_a^b f(t)\mathrm{d}t \approx \sum_{i=1}^{n} w_i \cdot f(t_i),$$

for $a \leq t_1 < \ldots < t_n \leq b$, $w_i > 0$.

A classical result:

Theorem

If we take $t_1, ..., t_n$ as the roots of the Legendre polynomial P_n and w_i as

$$w_i := \frac{2}{(1-t_i^2)\cdot[P_n'(t_i)]^2},$$

then

$$\int_{-1}^{1} q(t)dt = \sum_{i=1}^{n} w_i \cdot q(t_i)$$

for any polynomial q of degree lesser or equal to $2n-1$.

- A cubature formula, in the sense of last Theorem, can be interpreted as a discrete measure

$$\mu := \sum_{i=1}^{n} w_i \cdot \delta_{t_i},$$

for $a \leq t_1 < \ldots < t_n \leq b$, $w_i > 0$, such that

- $$\int_{-1}^{1} q(t)\mathrm{d}t = \int_{-1}^{1} q(t)\mathrm{d}\mu,$$

for any polynomial q with degree lesser or equal to $2n-1$.

- This condition is equivalent to claim

$$\int_{-1}^{1} m \; \mathrm{d}t = \int_{-1}^{1} m \; \mathrm{d}\mu,$$

for any monomial m with degree lesser or equal to $2n-1$.

Control Theory

- A control problem typically has the form

$$\min_{u(t)} \int_{[0,T]} h(t, x(t), u(t))\mathrm{d}t + H(x(T))$$

$$s.t$$

$$\frac{dx}{dt} = f(t, x(t), u(t)),$$

for $u(t), x(t) \in C^{\infty}([0, T])$, $x(t) \in X$, $u(t) \in U, x(0) = x_0$.

- In general are not so easy: proposed solutions depend of structure of functions $h(t, x, u)$, $H(x)$ and simplifications of dynamic $\frac{dx}{dt} = f(t, x(t), u(t))$.

Special case

- If both h, H are equal to zero and $f := f(t, x(t))$, control problem is a dynamical system.
- For example: height y of a ball falling has equations

$$\begin{pmatrix} \dot{y} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -g \end{pmatrix},$$

making substitution $\dot{y} = v$.

- Henrion suggests re-write the given control problem as

$$\min_{\substack{\mu \in M^+([0,T]\times X\times U),\\ \mu_F \in M^+(X)}} \int_{[0,T]\times X\times U} h(t,x,u)\mathrm{d}\mu + \int_X H(x)\mathrm{d}\mu_F$$

$$s.t$$

$$\int_X v(T,x)\mathrm{d}\mu_F v(0,x(0)) = \int_{[0,T]\times X\times U} (\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}) f(t,x,u)\mathrm{d}\mu,$$

for all $v \in C^1([0,T] \times X)$.

- Important issue: Any optimum measure μ has as support the curve $(t, x(t), u(t))$, $t \in [0, T]$.
- This is a problem with INFINITE restrictions!
- We can get a relaxation considering the functions v JUST as monomials up to degree d.

- Assume $m_1, \ldots, m_n$ the list of monomials with degree lesser or equal to some big enough d.
-
  - $h = \sum_{i=1}^{n} h_i \cdot m_i$,
  - $H = \sum_{i=1}^{n} H_i \cdot m_i$,
  - $f = \sum_{i=1}^{n} f_i \cdot m_i$,
  - $(\frac{\partial m_j}{\partial t} + \frac{\partial m_j}{\partial x}) f(t, x, u) = \sum_i F_{ij} \cdot y_i$,
  - $h_i, H_i, f_i, F_{ij} \in \mathbb{R}$.
- Consider $y_i := \int_{[0,T]\times X\times U} m_i \, \mathrm{d}\mu$ and $z_i := \int_X m_i \, \mathrm{d}\mu_F$.

- With relaxation considering just monomials the problem results

$$\min_{H_y(d),H_z(d)\succeq 0} \sum_i h_i \cdot y_i + \sum_i H_i \cdot z_i$$

$$s.t$$

$$z_j - m_j(0, x(0)) = \sum_i F_{ij} \cdot y_i,$$

- This problem is SEMIDEFINITE!!.
- We can get the values y_i(Lasserre's method of Moments) and from that we want to recover measures μ and μ_F, in order to see their support.

Results

In order to recover μ from the values y_i consider the problem

$$\min_{\Delta} \|\Delta\|_{TV} \ \text{s.t} \ \int \Phi_i \, d\Delta = y_i. \tag{1}$$

The seminal result

Theorem (Gamboa, De Castro)

If μ is a discrete positive measure supported at $X \subseteq [-1, 1]$, $|X| = k$, and

$$\int x^i \, d\mu = y_i, \quad i = 0, \ldots n,$$

then μ is the unique minimizer of Problem (1), for $2n \geq k$.

Our Result

Theorem

If μ is a discrete positive measure supported at $X \subseteq K \subseteq \mathbb{R}^n$,$K$ compact and $|X| = k$, and

$$\int \Phi \, d\mu = y_i,$$

where $\Phi_1, \ldots, \Phi_m$ are the monomials in n variables up to degree d, then μ is the unique minimizer of Problem (1), for $d \geq \max(2g(X), i(X))$

- $g(X), i(X)$ are numbers depending of the algebraic structure of X.

The main idea of the proof in Theorem 3 is the existence of a dual certificate:

Definition

A dual certificate is a polynomial P such that

- $0 \leq P(x) \leq 1$ for $x \in K$.
- $P(x) = 1$ iff $x \in X$.

A dual certificate $P(x)$ recognizes the support of μ!.

Algorithm: Semidefinite Relaxation

1. Find a dual certificate $P = 1 - \frac{H}{M}$ solving the problem:

$$\min_{H} \int_K H \, \mathrm{d}\mu$$

$$s.t$$

$$H = \sum h_i^2, \; deg(h_i) \leq d, \; H \neq 0.$$

2. Find support X of μ as the maximum points of P.
3. $\mu = \sum_{x \in X} c_x \cdot \delta_x$, so find the values c_x solving the linear system of equations

$$\sum_{x \in X} c_x \Phi_i(x) = y_i, \quad i = 1, ..., m.$$

If signal measurements $(y_1, ..., y_m)$ are contaminated with some unknown noise $\epsilon = (\epsilon_1, ..., \epsilon_m)$ but such that $\|\epsilon\| \leq \delta$, the following problem is proposed to recover μ:

$$\min_{\nu \in \mathcal{S}(K)} \|\nu\|_{\mathrm{TV}} : \left\| \left(\int_K \phi_i d\nu - y_i \right)_{i=0,...,m} \right\|_2 \leq \delta \tag{2}$$

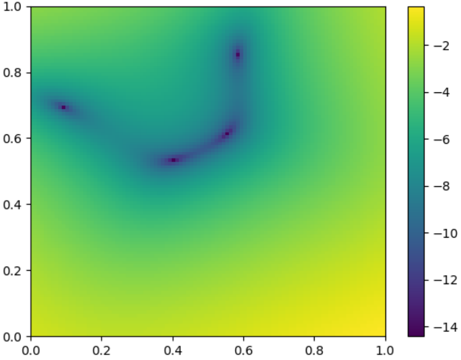
Theorem

Let $\hat{\Delta}$ be a discrete minimizer of (2) with V the vector space of polynomials up to degree d, $y_i := \int \phi_i d\mu + \epsilon_i$ and $\|(\epsilon_i)_i\|_2 \leq \delta$. If $d \geq 2g(X)$ and $\phi_0, \ldots, \phi_m$ are an orthonormal basis for V with respect to some probability measure on K then $\hat{\Delta}$ is "near" of μ in the following sense:

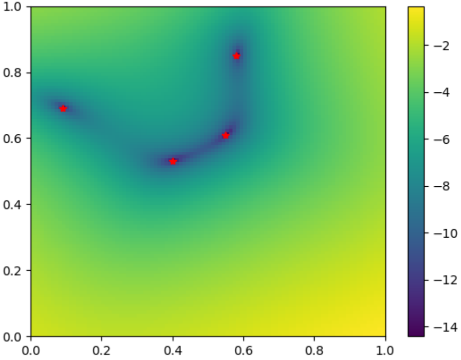
- *Assuming Y as the support of $\hat{\Delta}$, if $y \in Y$ is "near" of X, coefficient of δ_y is big.*
- *Otherwise its coefficient is little.*

Some Experiments

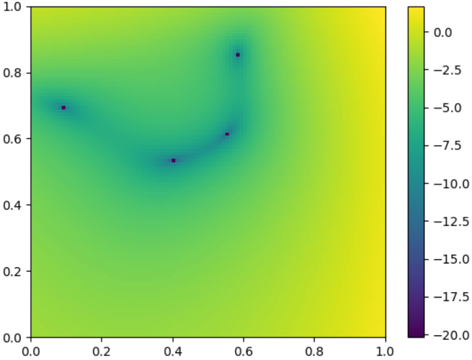
Measure with 4 support points



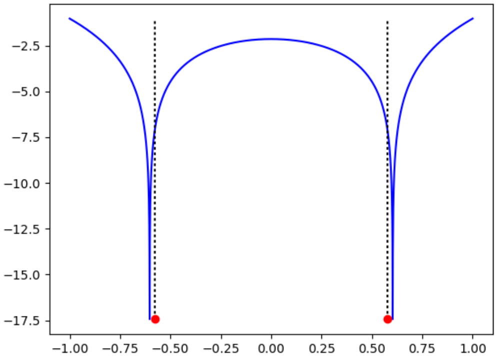
Measure with 4 support points



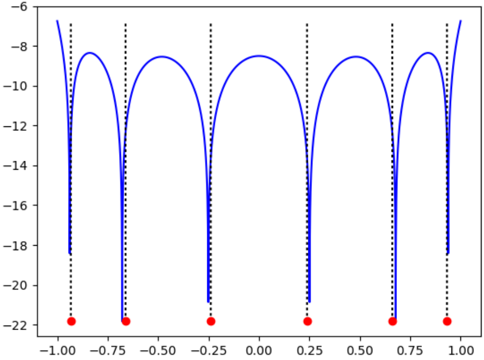
Measure with 4 support points



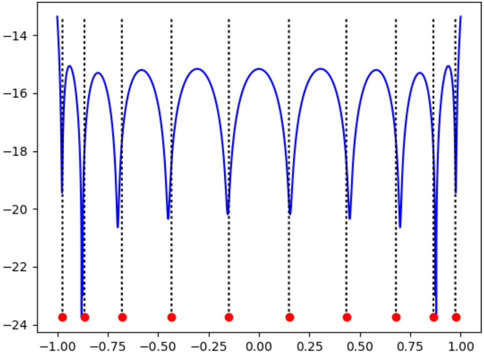
Zeroes of $H = M(P+1)$, $d = 2$



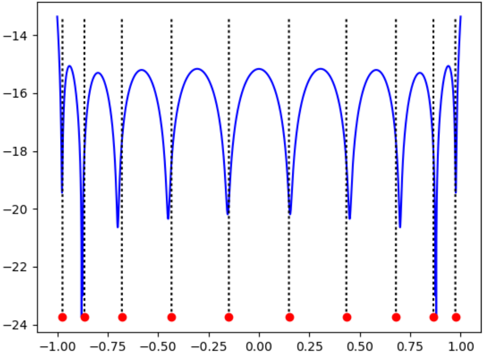
Zeroes of $H = M(P+1)$, $d = 6$



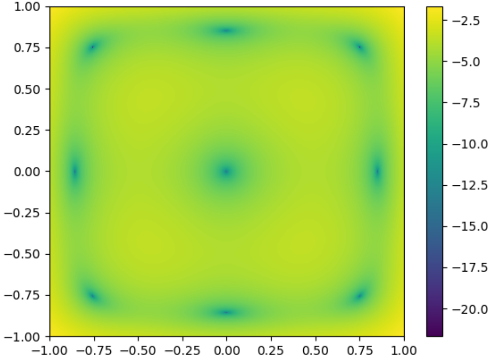
Zeroes of $H = M(P+1)$, $d = 10$



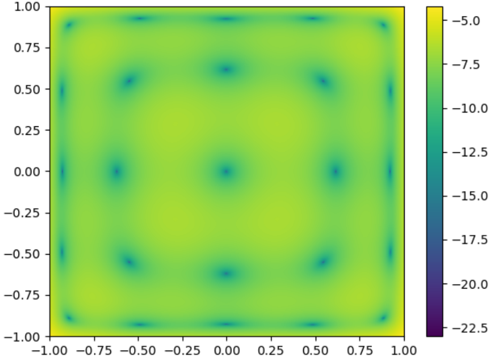
Zeroes of $H = M(P+1)$, $d = 10$



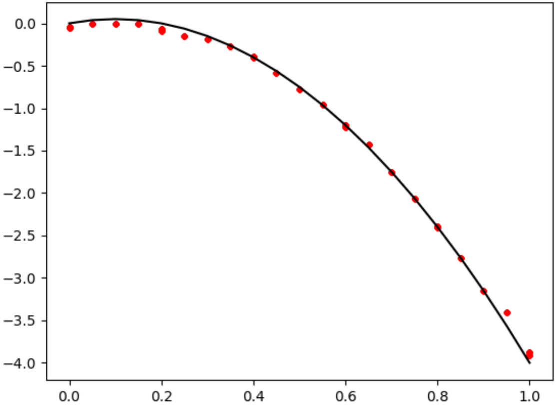
Zeroes of $H = M(P+1)$, $d = 3$



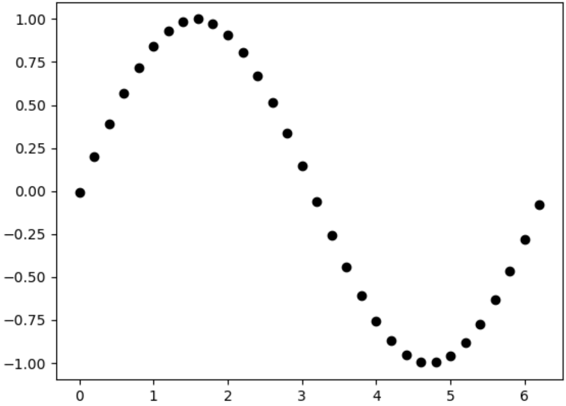
Zeroes of $H = M(P+1)$, $d = 5$



Ball Falling



Spring with mass



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