Matching markets:
An overview

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Quantil
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A Fruitful Decade for Matching Markets

- Matching market: Allocation of resources without monetary transfers.
- In the last decade there has been a lot of activity and excitement among economists working on matching markets.
- Theory, pioneered by Gale and Shapley (1962), matured to a point where matching theorists could make policy suggestions in key areas including education and health care.
- Until early 2000s the main practical application of matching theory was entry level labor markets such as the U.S. hospital-intern market.
- This trend has recently changed as matching theory found new applications in (often large scale) resource allocation problems of social importance.

A Fruitful Decade for Matching Markets

- Recently economists have been using economics to design institutions successfully, such as (1) labor markets where workers and firms are matched, (2) organizing organ donation network, and (3) student placement in schools.
  - Reforms of student assignment mechanisms in major school districts such as Boston, New York City, Chicago, Denver, New Orleans, and throughout England by all local authorities.
  - Establishment of regional and national centralized kidney exchange programs in the U.S., U.K., Sweden, and Turkey.
  - In a Congress testimony, Dr. Myron Gutmann (Assistant Director, SBE, NSF) emphasized that "research on matching markets has resulted in measurable gains for the U.S. taxpayer".

Labor Markets: The case of American hospital-intern markets.

- Medical students in many countries work as residents (interns) at hospitals.
- In the U.S. more than 20,000 medical students and 4,000 hospitals are matched through a clearinghouse, called NRMP (National Resident Matching Program).
- Doctors and hospitals submit preference rankings to the clearinghouse, and the clearinghouse uses a specified rule to decide who works where.
- Some markets succeeded while others failed. What is a “good way” to match doctors and hospitals?

Kidney Exchange

- There are close to 97,000 patients on the waiting list for cadaver kidneys in the U.S. as of June 2017.
- A staggering 5,000 people die every year waiting for a kidney transplant and another 5,000 are taken off the list because they are no longer healthy enough to receive a transplant.
- Most transplanted kidneys are from cadavers, but there are also substantial numbers of transplants from live donors.
  - 2-way exchange.
  - Cycles and chains.
  - 2 and 3 way exchanges and non-directed (altruistic) donor chains.
- How are an efficient and incentive-compatible system of exchanges organized, and what are its welfare implications?

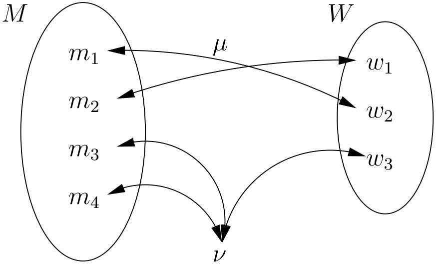
School Choice

- In many countries, children were automatically sent to a school in their neighborhoods.
- Recently, more and more cities in the United States and in other countries employ school choice programs: school authorities take into account preferences of children and their parents.
- Because school seats are limited (for popular schools), school districts should decide who is admitted.
- How should school districts decide placements of students in schools?

Matching Markets

Recipe for success: Discovery of important practical applications backed by solid theory.

Two-sided one-to-one matching problems: Marriage problem



- Let M be the set of men, W be the set of women, and $N = M \cup W$.
- Each man m has preference relation P_m over $W \cup \{\nu\}$.
- Each woman w has preference relation P_w over $M \cup \{\nu\}$.
- A problem is simply $P = (P_m, P_w)$.
- $\mathcal{M}(P)$: Set of possible matchings.
- $\varphi(P) \in \mathcal{M}(P)$ is a rule that recommends a matching.

Pareto–efficiency and the core

- For each $R \in \mathcal{R}$,
  μ is Pareto–efficient if there is no $\mu' \in \mathcal{M}(P)$ such that

  $$\text{for each } i \in N, \mu'(i) \; R_i \; \mu(i) \text{ and}$$

  $$\text{there is } i \in N, \mu'(i) \; P_i \; \mu(i).$$

- For each $R \in \mathcal{R}$,
  $\mu \in C(P)$ if there is no $S \subseteq N$ for which there is $\mu' \in \mathcal{M}(P)$ such that

  $$\mu'(S) = S,$$

  $$\text{for each } i \in S, \mu'(i) \; R_i \; \mu(i), \text{ and}$$

  $$\text{there is } i \in S, \mu'(i) \; P_i \; \mu(i).$$

Stability

- Individually rational: For each $i \in N$, $\mu(i)\ R_i\ \nu$.

A pair of different agents $\{m, w\}$ blocks μ if $w\ P_m\ \mu(m)$ and $m\ P_w\ \mu(w)$.

- Stable for P: Given P, μ is individually rational and there is no blocking pair.
- S(P) is the set of stable matchings.

Strategy–proofness

A rule φ is strategy–proof

if for each P, each $i \in M \cup W$, and each $\underbrace{P'_i}_{lie}$ such that

$$\varphi(\underbrace{P_i}_{truth}, P_{-i})(i) \; R_i \; \varphi(\underbrace{P'_i}_{lie}, P_{-i})(i).$$

The *deferred acceptance* algorithm

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

The *deferred acceptance* algorithm

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

Stage 1:

w_1	m_1, m_4, m_5	w_1	$[m_1]$
w_2		w_2	
w_3		w_3	
w_4	m_2, m_3	w_4	$[m_2]$

The *deferred acceptance* algorithm

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

Stage 2:

w_1	$[m_1]$
w_2	m_5
w_3	m_3
w_4	$[m_2], m_4$

w_1	$[m_1]$
w_2	$[m_5]$
w_3	$[m_3]$
w_4	$[m_4]$

The *deferred acceptance* algorithm

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

Stage 3:

w_1	$[m_1]$
w_2	$[m_5], m_2$
w_3	$[m_3]$
w_4	$[m_4]$

w_1	$[m_1]$
w_2	$[m_2]$
w_3	$[m_3]$
w_4	$[m_4], m_5$

The *deferred acceptance* algorithm

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

The men proposing deferred acceptance algorithm produces the matching:

w_1	$[m_1]$
w_2	$[m_2]$
w_3	$[m_3]$
w_4	$[m_4]$
m_5	$[m_5]$

The *deferred acceptance* algorithm

- Men proposing:

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

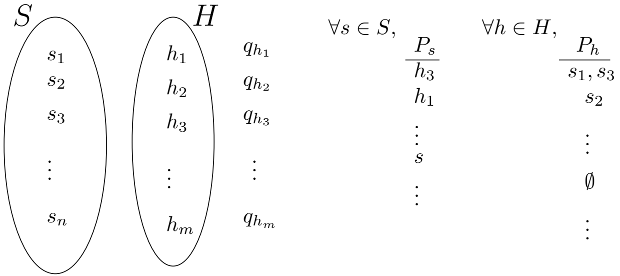
- Women proposing:

P_{m_1}	P_{m_2}	P_{m_3}	P_{m_4}	P_{m_5}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_1	w_4	w_4	w_1	w_1	m_2	m_3	m_5	m_1
w_2	w_2	w_3	w_4	w_2	m_3	m_1	m_4	m_4
w_3	w_3	w_1	w_3	w_4	m_1	m_2	m_1	m_5
w_4	w_1	w_2	w_2	m_5	m_4	m_4	m_2	m_2
					m_5	m_5	m_3	m_3

Results

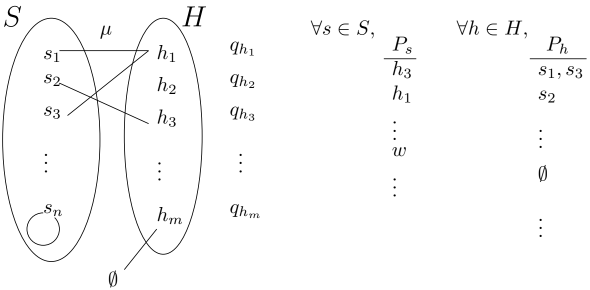
- $S(P) = C(P)$.
- $S(P) \subseteq PE(P)$.
- Men (women)–proposing deferred acceptance algorithm produces a men (women)-optimal stable matching, i.e. a stable matching that every men (women) likes at least as well as any other stable matching.
- There is no stable and strategy–proof rule.
- Men (women)–proposing deferred acceptance rule is strategy–proof for men (women).

Two-sided one-to-one matching problems: Medical residency market



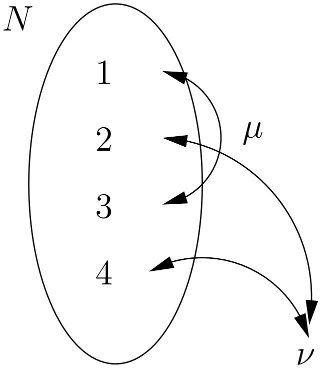
- Let S be the set of medical students, H be the set of hospitals.
- Each student $s \in S$ has a preference relation P_s over the hospitals and the prospect of "being unmatched."
- For each hospital h, there is a quota $q_h \geq 1$.
- A subset of students $S' \subseteq S$ is feasible for hospital h if $|S'| \leq q_h$.
- Let $\mathcal{F}(S, q_h) = \{S' \subseteq S : |S'| \leq q_h\}$ denote the collection of feasible subsets of students for hospital h.
- Each hospital h has a preference relation P_h over $\mathcal{F}(S, q_h)$ that satisfies "responsiveness".

Two-sided one-to-one matching problems: Medical residency market



- A matching is a correspondence μ on the set $S \cup H$ such that
  - for all $s \in S$, either $\mu(s) \in H$ or $\mu(s) = s$,
  - for all $h \in H$, $\mu(h) \in \mathcal{F}(S, q_h)$, and
  - for all $s \in S$ and $h \in H$, $\mu(s) = h \iff s \in \mu(h)$.
- A rule φ assigns a matching to each problem (P_S, P_H, q_H).
- Most of the results in two–sided one–to–one matching markets are extended simply to two–sided many–to–one matching markets.

One-sided one-to-one matching problems: Roommate problem



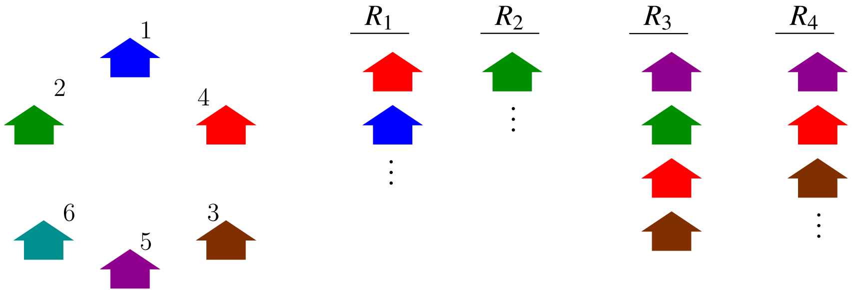
- $\mathcal{M}(P)$: Set of possible matchings.
- $\varphi(P) \in \mathcal{M}(P)$ is a rule that recommends a matching.

The core might be empty

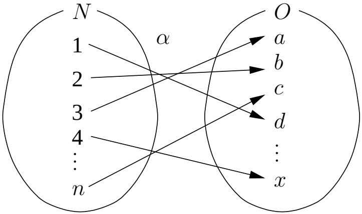
P_1	P_2	P_3
3	1	2
2	3	1
1	2	3

One-sided matching problems with endowments: House exchange

Each person owns one object and needs one object.



One-sided matching problems with endowments: House exchange



R_i
a
g
b
c
d
e
$\vdots$

- Preferences for $i \in N$ over O: $R_i \in \mathcal{R}$.
- Allocations: $A \equiv \{\text{all bijections } \alpha : N \to O\}$.
- Endowment: $\omega \in A$.
- A single-valued rule $\varphi : \mathcal{R}^N \times A \to A$.

Pareto–efficiency

For each (R, ω),

α is Pareto–efficient if there is no $\alpha' \in \mathcal{A}$ such that:

for each $i \in S$, $\alpha'(i)\ R_i\ \alpha(i)$ and

there is $i \in S$, $\alpha'(i)\ P_i\ \alpha(i)$.

Core

For each (R, ω),

$\alpha \in C(R, \omega)$ if there are no $S \subseteq N$ and no $\alpha' \in \mathcal{A}$ such that:

$\alpha'(S) = S$,

for each $i \in S, \alpha'(i) \; R_i \; \alpha(i),$ and

there is $i \in S, \alpha'(i) \; P_i \; \alpha(i)$.

Individual rationality

For each (R, ω)

α is individually rational if for each $i \in N$,

$$\alpha(i) \; R_i \; \omega(i).$$

Strategy–proofness

A rule φ is strategy–proof

if for each (R, ω), each $i \in N$, and each $\underbrace{R'_i}_{lie}$ such that

$$\varphi(\underbrace{R_i}_{truth}, R_{-i})(i) \; R_i \; \varphi(\underbrace{R'_i}_{lie}, R_{-i})(i).$$

The *top trading cycles* algorithm

Gale's top trading cycles (TTC) algorithm:

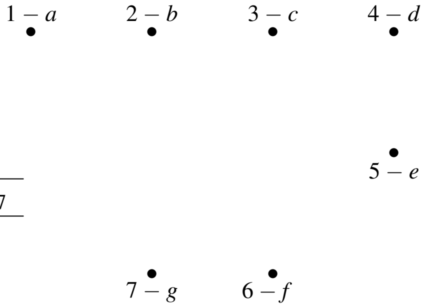
- Step 1: Let each agent point to her top choice house and each house point to its owner. In this graph there is necessarily a cycle and no two cycles intersect. Remove all cycles from the problem by assigning each agent the house that she is pointing to.
- ...
- Step k: Let each remaining agent point to her top choice among the remaining houses and each remaining house point to its owner (note that houses leave with their owners and owners leave with their houses, so a house remaining in the problem implies that the owner is still in the problem and vice versa). There is necessarily a cycle and no two cycles intersect. Remove all cycles from the problem by assigning each agent the house that she is pointing to.
- The algorithm terminates when no agents and houses remain. The assignments formed during the execution of the algorithm is the matching outcome.

The *top trading cycles* algorithm

Let $\omega = (a, b, c, d, e, f, g)$

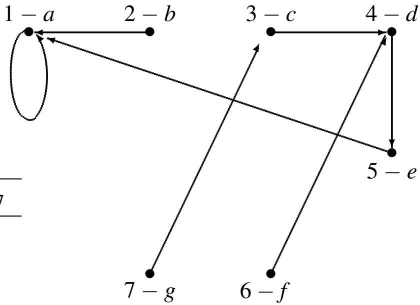
R_1	R_2	R_3	R_4	R_5	R_6	R_7
$\textcircled{a}$	a	d	e	a	d	c
$\vdots$	$\textcircled{b}$	f	$\textcircled{d}$	c	g	f
	$\vdots$	$\textcircled{c}$	$\vdots$	$\textcircled{e}$	$\textcircled{f}$	$\textcircled{g}$

The *top trading cycles* algorithm



R_1	R_2	R_3	R_4	R_5	R_6	R_7
a	a	d	e	a	d	c
⋮	b	f	d	c	g	f
	⋮	c	⋮	e	f	g

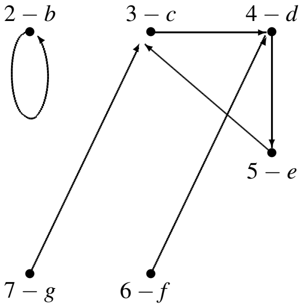
The *top trading cycles* algorithm



R_1	R_2	R_3	R_4	R_5	R_6	R_7
(a)	a	d	e	a	d	c
$\vdots$	(b)	f	(d)	c	g	f
	$\vdots$	(c)	$\vdots$	(e)	(f)	(g)

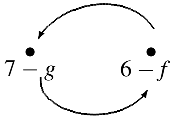
The *top trading cycles* algorithm

R_1	R_2	R_3	R_4	R_5	R_6	R_7
ⓐ	a	d	e	a	d	c
⋮	ⓑ	f	ⓓ	c	g	f
	⋮	ⓒ	⋮	ⓔ	ⓕ	ⓖ



The *top trading cycles* algorithm

R_1	R_2	R_3	R_4	R_5	R_6	R_7
ⓐ	a	d	e	a	d	c
⋮	ⓑ	f	ⓓ	c	g	f
	⋮	ⓒ	⋮	ⓔ	ⓕ	ⓖ



The *top trading cycles* algorithm

Let $\omega = (a, b, c, d, e, f, g)$

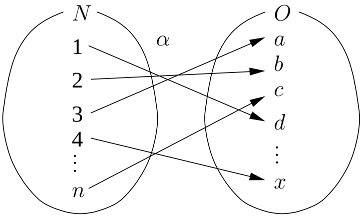
R_1	R_2	R_3	R_4	R_5	R_6	R_7
a	a	d	e	a	d	c
$\vdots$	b	f	d	c	g	f
	$\vdots$	c	$\vdots$	e	f	g

Results

For each (R, ω),

- $C(R, \omega) \neq \emptyset$.
- $|C(R, \omega)| = 1$ if the preferences are strict.
- Top trading cycles algorithm achieves a core matching. Hence, it is Pareto–efficient and individually rational.
- It is also sustainable by a competitive equilibrium.
- The top trading cycle rule is strategy-proof.
- This model can be extended to situation where existing house-owners and new entrants coexist (House allocation with existing owners).
- A variation of the house exchange model to represent the kidney–exchange market.

One-sided matching problems without endowments: House allocation



R_i
a
g
b
c
d
e
$\vdots$

- Preferences for $i \in N$ over O: $R_i \in \mathcal{R}$.
- Allocations: $A \equiv \{\text{all bijections } \alpha : N \to O\}$.
- A single-valued rule $\varphi : \mathcal{R}^N \to A$.

The *serial dictatorship* algorithms

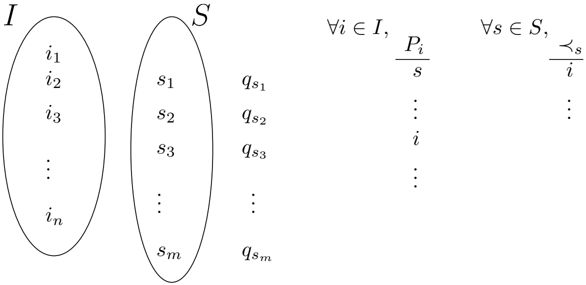
- Step 0: Fix a rank ordering σ over the set of agents.
- Step 1: Assign $\sigma(1)$ her most preferred house.

  In general, for any t = 1, 2, ...
- Step t: Assign $\sigma(t)$ her most preferred remaining house.
- The algorithm terminates when there is no agent or house left. If there are still agents left, then they are not assigned a house.

Results

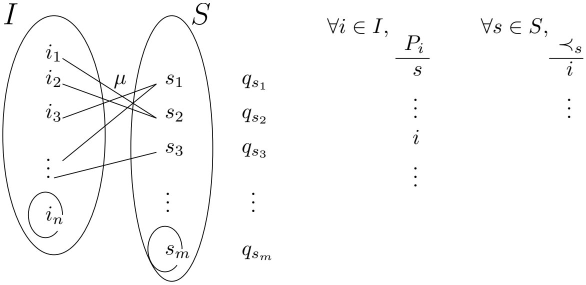
- Serial dictatorship rules are Pareto–efficient.
- Serial dictatorship rules are strategy–proof.
- We can define Step 0 : Pick a rank ordering σ uniformly at random from the set of all rank orderings.
- Hence, a rule is $\varphi : \mathcal{R}^N \to \Delta A$ and one can define random serial dictatorship rules.
- These rules are ex–post Pareto–efficient and strategy–proof.
- There is an incompatibility between ex–ante Pareto–efficiency, strategy–proofness, and fairness, i.e., two agents with the same preferences receive the same random allocation to each other.

School choice problem



- a set of students $I = \{i_1, \ldots, i_n\}$,
- a set of schools $S = \{s_1, \ldots, s_m\}$,
- a capacity vector $q = (q_{s_1}, q_{s_2}, \ldots, q_{s_m})$,
- a list of strict student preferences $P = (P_{i_1}, \ldots, P_{i_n})$ and
- a list of strict school priorities $\prec = (\prec_{s_1}, \ldots, \prec_{s_m})$.
- A problem $(P, \prec, q)$, P for simplicity.

Assignments and rules



- An assignment is a correspondence μ on the set $I \cup S$ such that
  - for each $i \in I$, either $\mu(i) \in S$ or $\mu(i) = i$,
  - for each $s \in S$, either $\mu(s) \subset I$ or $\mu(s) = s$,
  - for each $s \in S$, $|\mu(s)| \leq q_s$, and
  - for each $i \in I$ and $s \in S$, $\mu(i) = s \iff i \in \mu(s)$.
- For each P, let $A(P)$ be the set of assignments.
- A rule is a single-valued function, $\varphi(P) \in A(P)$.

Pareto–efficiency

- μ is Pareto–efficient if there is no $\mu' \in A(P)$ such that for each $i \in N$, $\mu'(i)\ R_i\ \mu(i)$ and there is $i' \in N$ such that $\mu'(i')\ P_i\ \mu(i')$.
- Set of Pareto–efficient assignments $E(P)$.

Stability

- μ is non-wasteful if there is no student i and a school s such that
  1. $s \; P_i \; \mu(i)$ and
  2. $|\mu(s)| < q_s$.
- μ is individually rational if for all $i \in I$,

$$\mu(i) \; R_i \; i.$$

- student i has justified envy at μ if there is a school s such that
  1. $s \; P_i \; \mu(i)$ and
  2. $\exists j \in \mu(s)$ such that $i \prec_s j$.

Stability

- μ stable if it is *individually rational*, *non-wasteful*, and no student has *justified envy*.
- Set of stable assignments $\Sigma(P)$.

Strategy–proofness

A rule φ is strategy–proof

if for each $(P, \prec, q)$, each $i \in I$, and each $\underbrace{P_i'}_{lie}$ such that

$$\varphi(\underbrace{P_i}_{truth}, P_{-i})(i) \; R_i \; \varphi(\underbrace{P_i'}_{lie}, P_{-i})(i).$$

The *immediate acceptance* algorithm [a.k.a. Boston mechanism]

- Step 1: In this step, only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there is no seat left or there is no student left who has listed it as his first choice.

  In general, for any t = 1, 2, ...

- Step t: In this step, only the *t*-th choices of the students are considered. For each school, consider the students who have listed it as their *t*-th choice and assign seats of the school to these students one at a time following their priority order until either there is no seat left or there is no student left who has listed it as his *t*-th choice.

The *deferred acceptance* algorithm

- Step 1: Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

  In general, for any t = 1, 2, ...
- Step t: Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.
- The algorithm terminates when no student proposal is rejected and each student is assigned her final tentative assignment

The *top trading cycles* algorithm

- Step 1: Assign a counter for each school which keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools. Each student points to her favorite school under her announced preferences. Each school points to the student who has the highest priority for the school. Since the number of students and schools are finite, there is at least one cycle. Moreover, each school can be part of at most one cycle. Similarly, each student can be part of at most one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in cycle is reduced by one and if it reduces to zero, the school is also removed. The counters of the schools not in a cycle remain the same.

  In general, for any t = 1, 2, ...
- Step t: Each remaining student points to her favorite school among the remaining schools and each remaining school points to the student with highest priority among the remaining students. There is at least one-cycle. Every student in a cycle is assigned a seat at the school that she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero the school is also removed.

Results

	IA	DA	TTC
Pareto–efficiency	+	−	+
Stability	−	+	−
Strategy–proofness	−	+	+

- There is an incompatibility between Paerto–efficiency, stability, and strategy–proofness.
- In Boston, the immediate acceptance rule was originally implemented in July, 1999 but was abandoned in 2005 and replaced by the deferred acceptance rule.
- The school district orders students by priority block. Within each block, students are ordered via a lottery system. How to break the ties is also very important.

Mixture of models

- Allocation of not “convex” resources (not money).
- Preferences? Priorities? a mix?
- Año rural en Colombia: mixture of two sided matching and school choice.
- Coming soon: In what are we working?

Thank you