Discrete conic quadratic optimization: strong formulations and algorithms

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Research overview

- Fixed-charge constraints[1] [2]. *Published in Networks*
  Simultaneous study of two fixed-charge constraints
- Maximizing utility functions[1]. *Forthcoming in Operations Research*
  Approximation algorithm for nonlinear maximization
- Simplex for conic quadratic minimization[1]. *In preparation*
  Algorithm for subproblems in branch & bound
- Study of conic quadratic constraints[1]. *Submitted*
  Valid inequalities for conic quadratic optimization

[1] Joint work with Alper Atamtürk
[2] Joint work with Simge Küçükyavuz

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Agenda

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1. Maximizing utility functions
   - Problem statement
   - Literature
   - Approximation algorithm
   - Computations
2. Study of conic quadratic constraints
3. Conclusions & future work

Problem statement

- $g : \mathbb{R}_+ \to \mathbb{R}_+$: monotone concave function
- $a, c \in \mathbb{R}_+^n$: objective vectors

$$\max\; a'x + g(c'x)$$
$$\text{s.t. } x \text{ is a vertex of a polytope}$$

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- $\max_{x \text{ vertex}} \; a'x \to$ linear optimization

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- $\max\limits_{x \text{ vertex}} \quad a'x \to$ linear optimization
- $\max\limits_{x \text{ in polytope}} \quad a'x + g(c'x) \to$ convex optimization

Project Evaluation and Review Technique (PERT)

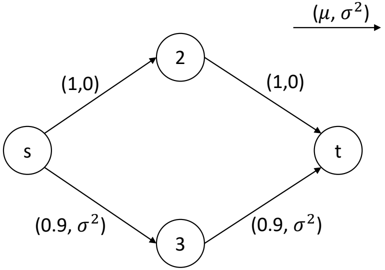
- Project composed of many individual activities
- Each activity requires a given amount of time to complete
- Precedence relations between activities

Interesting questions

- What is the duration of the project?
- Which are the critical activities?
  - Activities more likely to cause delays in the project
  - Activities that should be monitored carefully
  - Activities where additional resources should be allocated

Project Evaluation and Review Technique (PERT)

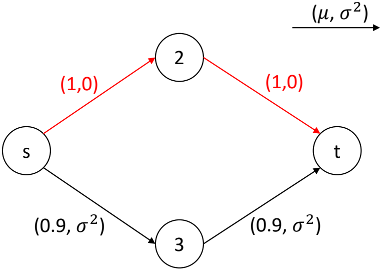
- A: set of activities required to complete a project
- G: DAG encoding precedence relations
- $\tilde{d}_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$: duration of activity $i \in A$ (independent)



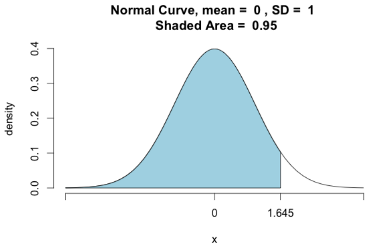
Project Evaluation and Review Technique (PERT)

Traditional PERT (e.g. Nahmias 2001, Chapter 9)

- Find path with largest expected duration
- Estimate project duration using this path



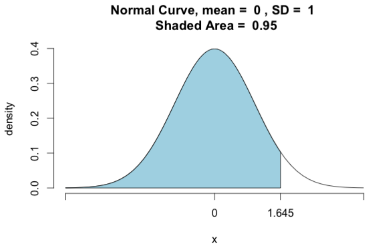
Value-at-Risk



For $\tilde{d} \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathrm{VaR}_\alpha(\tilde{d}) := \sup \left\{ k \in \mathbb{R} : \Pr(\tilde{d} \leq k) \leq \alpha \right\}$$

Value-at-Risk



For $\tilde{d} \sim \mathcal{N}(\mu, \sigma^2)$

$$\begin{aligned}\mathrm{VaR}_\alpha(\tilde{d}) :=& \sup\left\{k \in \mathbb{R} : \Pr(\tilde{d} \leq k) \leq \alpha\right\} \\ =& \mu + \Phi^{-1}(\alpha)\sigma\end{aligned}$$

Φ: CDF of the standard normal distribution

VaR_α-critical path

Path with maximum Value-at-Risk (VaR) at confidence level α

- x_a: 1 if activity $a \in A$ belongs to the critical path, 0 otherwise
- $\delta^+(i), \delta^-(i)$: incoming and outgoing arcs to node i

$$\max \sum_{a \in A} \mu_a x_a + \Phi^{-1}(\alpha) \sqrt{\sum_{a \in A} \sigma_a^2 {x_a}^2}$$

$$\text{s.t.} \quad \sum_{a \in \delta^+(i)} x_a - \sum_{a \in \delta^-(i)} x_a = \begin{cases} -1 & \text{if } i = s \\ 1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$$

$$x_a \in \{0, 1\}, \quad \forall a \in A$$

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s.t. x is a vertex of the path polytope

Assortment optimization

- N: set of ads
- M: set of advertising spots
- a_{ij}: profit from displaying ad $i \in N$ in spot $j \in M$
- β: fixed profit per click
- **Goal:** choose assortment $S \subseteq N \times M$ maximizing profit

  Assortment: assignment of ads to advertising spots

Assortment optimization

Multinomial logit (MNL) discrete choice model

- Customers are utility maximizers
- μ_{ij}: *estimated* customer utility for ad i in spot j
- No-click choice has zero estimated utility
- Probability of clicking ad i in spot j given assortment $S \subseteq N \times M$

$$P_{ij}(S) = \frac{e^{\mu_{ij}}}{1 + \sum_{(k,\ell)\in S} e^{\mu_{k\ell}}}$$

Assortment optimization

x_{ij}: 1 if ad i is assigned to spot j, 0 otherwise

$$\begin{aligned}
\max \quad & \sum_{i \in N} \sum_{j \in M} a_{ij} x_{ij} + \beta \frac{\sum_{i \in N} \sum_{j \in M} e^{\mu_{ij}} x_{ij}}{1 + \sum_{i \in N} \sum_{j \in M} e^{\mu_{ij}} x_{ij}} \\
\text{s.t.} \quad & \sum_{i \in N} x_{ij} \leq 1, \quad \forall j \in M \qquad \text{(spots contain at most one ad)} \\
& \sum_{j \in M} x_{ij} \leq 1, \quad \forall i \in N \qquad \text{(ads are displayed at most once)} \\
& x_{ij} \in \{0, 1\}, \quad \forall i \in N, \forall j \in M
\end{aligned}$$

Assortment optimization

x_{ij}: 1 if ad i is assigned to spot j, 0 otherwise

$$\begin{aligned} \max \ & \sum_{i\in N}\sum_{j\in M} a_{ij}x_{ij} + \beta \frac{\sum_{i\in N}\sum_{j\in M} e^{\mu_{ij}}x_{ij}}{1+\sum_{i\in N}\sum_{j\in M} e^{\mu_{ij}}x_{ij}} \\ \text{s.t.} \ & \sum_{i\in N} x_{ij} \le 1, \quad \forall j\in M \qquad \text{(spots contain at most one ad)} \\ & \sum_{j\in M} x_{ij} \le 1, \quad \forall i\in N \qquad \text{(ads are displayed at most once)} \\ & x_{ij}\in\{0,1\}, \quad \forall i\in N, \forall j\in M \end{aligned}$$

- $c_{ij} := e^{\mu_{ij}}$, $g(c'x) = \beta\frac{c'x}{1+c'x}$
- Feasible region is the set of vertices of assignment polytope

Other applications

- Reliability of a parallel system

  Maximize tradeoff between benefit and reliability

- Robust conic quadratic optimization

  Minimize Value-at-Risk with discrete uncertainty set

- Multi-armed bandit

  Exponential number of arms characterized by combinatorial set

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Literature

- X: polytope
- V_X: vertices of X

$$\max_{x \in V_X} f(x) := a'x + g(c'x)$$

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Literature

- X: polytope
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$$\max_{x \in V_X} f(x) := a'x + g(c'x)$$

- Maximization of a submodular function
- **Greedy algorithm**
  - 0.63-approx. for cardinality constraint (Nemhauser et al. 1978)
  - 0.5-approx. for matroid polytopes (Fisher et al. 1978)

- **Other approaches**
  - 0.63-approx. for matroids (Calinescu et al. 2011). Complexity: $\tilde{O}(n^8)$
  - $(0.63 - \epsilon)$-approx. for down-monotone polytopes (Chekuri et al. 2014). Complexity depends on X, requires ellipsoid method

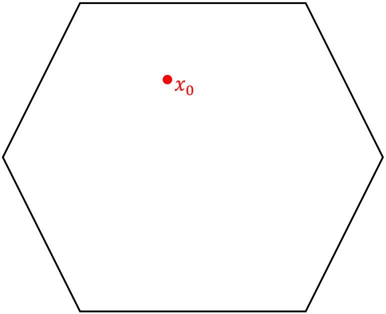
Agenda

Approximation algorithm

Algorithm

1. Find $x_0 \in \arg\max_{x \in X} a'x + g(c'x)$

Feasible region



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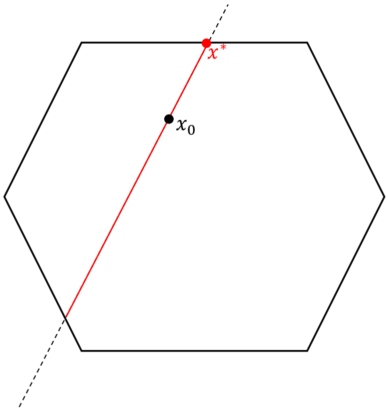
2. x^*: optimal extreme point to

$$\begin{aligned} &\max a'x \\ &\text{s.t. } c'x = c'x_0 \\ &x \in X \end{aligned}$$

Feasible region



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Prop: x^* lies on an edge of X

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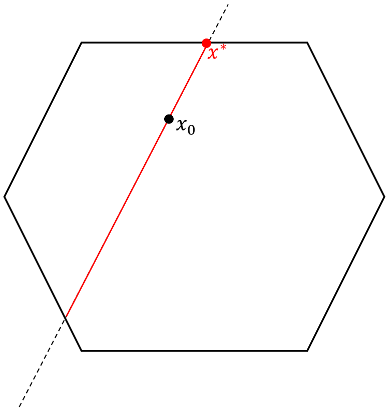

Approximation algorithm

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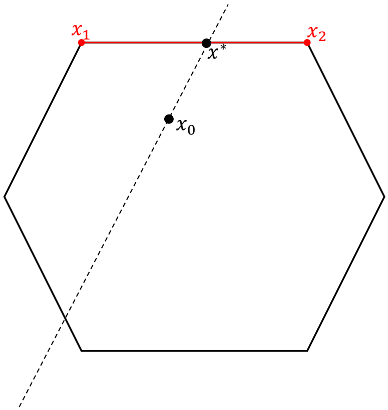
Prop: x^* lies on an edge of X

3. Two vertices x_1 and x_2 on edge
4. **return** $\arg\max_{x_i} f(x_i)$

Feasible region



Approximation ratio

- x_1: best vertex solution
- x^*: optimal solution of relaxation
  $\implies$ $f(x^*)$ is an upper bound on the optimal value

Optimality gap:

$$\Delta = \frac{f(x^*) - f(x_1)}{f(x_1)}$$

Approximation ratio:

$$\frac{1}{1+\Delta}$$

Approximation ratios

Proposition

If g is the square root function, then the algorithm is a 0.8-approximation

- Improvement over previous 0.63-approximations (for simple regions)
- Valid for *any* polytope (e.g. paths)

Approximation ratios

Proposition

If g is the square root function, then the algorithm is a 0.8-approximation

- Improvement over previous 0.63-approximations (for simple regions)
- Valid for *any* polytope (e.g. paths)

Proposition

If g is monotone, then the algorithm is a 0.5-approximation

- Same ratio as greedy algorithm (for matroid polytopes)
- Valid for *any* polytope (e.g. assignment)

Agenda

PERT

- `Det`: traditional PERT method. Complexity: $O(|A|)$
- `VaR`: VaR critical path. Complexity: $O(|A| \log \frac{1}{\epsilon})$
- Measure the error between estimated VaR and true VaR of the project
- Averages of 15 instances, $200 \leq |A| \leq 1000$

Confidence level (%)	Error (%) Det	Error (%) VaR	Improvement (%)
90.0	31.7	22.7	28.4
97.5	25.7	12.6	51.0
99.0	24.6	10.6	56.9

Solution times < 1s

Assortment optimization

- $n \times n$ assignments
- Approximation algorithm complexity $O(n^3 \log \frac{1}{\epsilon})$
- Optimality gap $\Delta = \frac{f^*_{\text{relax}} - f^*_{\text{vertex}}}{f^*_{\text{vertex}}}$
- Averages over five instances

n	3	10	100	1000
Time (ms)	1	2	66	2,892
Δ (%)	0.24	0.01	0.00	0.00

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Problem statement

- $P \subseteq \{0,1\}^n \times \mathbb{R}^m$: feasible region
- $\tilde{u}$: costs (uncertain)
  - $\tilde{u} \sim \mathcal{N}(a, c)$, or
  - $\tilde{u} \in \left\{ u \in \mathbb{R}^{n+m} : \sum_{i=1}^{n+m} \frac{(u_i - a_i)^2}{c_i} \leq 1 \right\}$

Minimize "worst case" scenario

$$\min\ a'x + \sqrt{\sum_{i=1}^{n+m} c_i x_i^2}$$

$$\text{s.t. } x \in P$$

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Minimize "worst case" scenario

$$\min\ a'x + z$$

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$$x \in P, z \in \mathbb{R}_+$$

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Pure-binary case

$$K = \left\{ (x, z) \in \{0, 1\}^n \times \mathbb{R}_+ : \sqrt{\sum_{i=1}^{n} c_i x_i^2} \leq z \right\}$$

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$$K = \left\{ (x, z) \in \{0,1\}^n \times \mathbb{R}_+ : \sqrt{c'x} \leq z \right\}$$

- $f : \{0,1\}^n \to \mathbb{R}$ with $f(x) = \sqrt{c'x}$ is submodular
- M is polymatroid associated with f

$$M := \left\{ \pi \in \mathbb{R}^n_+ : \pi' x \leq f(x), \ \forall x \in \{0,1\}^n \right\}$$

- Π: set of extreme points of M
- Edmonds (1970) gave characterization of Π

Pure-binary case

Theorem (Atamtürk and Narayanan 2008)

$$conv(K) = \begin{cases} \pi' x \leq z & \forall \pi \in \Pi \\ 0 \leq x_i \leq 1 & i = 1, \ldots, n \end{cases}$$

Pure-binary case

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Separation problem: given fractional $\bar{x}$, find best inequality to add:

$$\max_{\pi \in \Pi} \pi' \bar{x}$$

Can be solved in $O(n \log n)$ (Edmonds 1970)

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- Pure-binary case completely understood
- Mixed-binary case poorly understood

Conic constraints with continuous variables

$$H = \left\{ (x, y, z) \in \{0,1\}^n \times \mathbb{R}_+^{m+1} : \sqrt{c'x + \sum_{i=1}^{m} d_i y_i^2} \leq z \right\}$$

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- Polymatroid inequalities:

$$\sqrt{(\pi' x)^2 + \sum_{i=1}^{m} d_i y_i^2} \leq z, \quad \forall \pi \in \Pi$$

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Theorem (Convex hull of H)

Inequalities

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and bound constraints completely describe conv(H)

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Other contributions

- Bounded continuous variables

Conic constraints with continuous variables

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and bound constraints completely describe conv(H)

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Other contributions

- Bounded continuous variables
- Exploit additional constraints to get stronger inequalities

Agenda

General covariance matrix

- $C \succeq 0$: covariance matrix

$$\min \; a'x + \sqrt{x'Cx}$$

$$\text{s.t.} \; \sum_{i=1}^{n} x_i = k$$

$$x \in \{0,1\}^n$$

General covariance matrix

- $C \succeq 0$: covariance matrix

$$\min \; a'x + \sqrt{\sum_{i=1}^{n} D_{ii} x_i + x' F x}$$

$$\text{s.t.} \; \sum_{i=1}^{n} x_i = k$$

$$x \in \{0,1\}^n$$

1. Decompose $C := D + F$
   - $D \succeq 0, F \succeq 0$
   - D is diagonal

General covariance matrix

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$$\sqrt{x'Fx} \leq y$$

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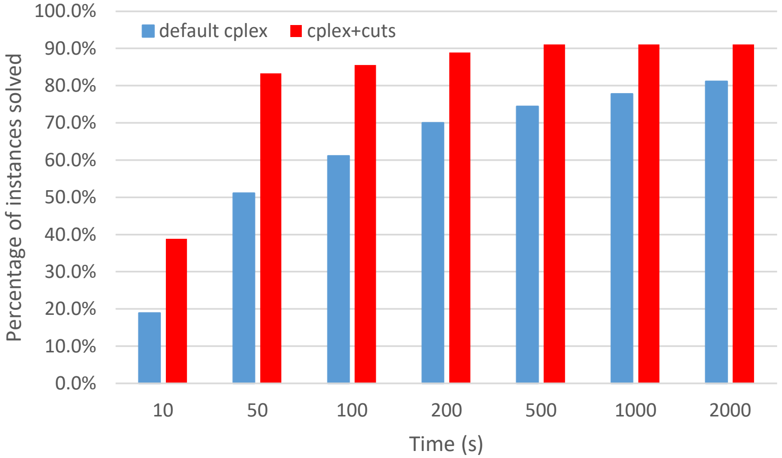
Computational experiments - General covariance

- $n = 200$
- CPLEX 12.6.2, two hour time limit, average over 15 instances

diag	k	default cplex rimp	default cplex time	default cplex egap	cplex+cuts rimp	cplex+cuts time	cplex+cuts egap	speedup
low	20	19.7	720	0.0	55.6	109	0.0	$\times 7$
	30	12.0	399	0.0	56.2	18	0.0	$\times 22$
	40	18.9	187	0.0	54.2	9	0.0	$\times 21$
high	20	14.5	4,034	0.9	51.0	2,650	0.2	$\times 2+$
	30	13.4	2,014	0.2	74.8	187	0.0	$\times 10+$
	40	21.8	1,128	0.1	74.0	15	0.0	$\times 75+$
Average		16.7	1,414	0.2	61.0	498	0.0	

Computational experiments - General covariance



Assortment optimization with mixed-MNL

- N: set of products
- M: set of customer classes
- $c_j, d_j \in \mathbb{R}_+^N$: parameters of MNL corresponding to class $j \in M$

$$\max \sum_{j \in M} \frac{c_j' x}{1 + d_j' x}$$
$$\text{s.t.} \sum_{i \in N} x_i \leq k$$
$$x \in \{0,1\}^N$$

- $\mathcal{N}P$-hard
- Hard to approximate with algorithm polynomial in $|M|$
- Can be put in conic form

Computational experiments

- $|N| = 200$
- Classic MILP formulation vs. conic formulation with cuts
- CPLEX 12.6.2, two hour limit, averages over five instances

$\|M\|$	k	cplex(MILP)			cplex(conic)+cuts			speedup
		rimp	time	egap	rimp	time	egap	
4	10	0.0	7,200	25.9	100.0	3	0.0	×2,400+
	20	0.0	7,200	14.7	100.0	8	0.0	×900+
	40	0.0	7,200	2.7	100.0	81	0.0	×90+
20	10	0.0	7,200	45.2	99.9	130	0.0	×55+
	20	0.0	7,200	18.3	100.0	159	0.0	×45+
	40	0.0	7,200	3.1	100.0	132	0.0	×55+
Average		0.0	7,200	18.3	100.0	86	0.0	

Agenda

Conclusions

- Nonlinear utility functions arise when modeling uncertainty
- Current approaches for discrete nonlinear optimization are limited
- Possible to have efficient algorithms

Future work

Extensions of polymatroid inequalities

- Polymatroid inequalities are “strong” for some feasible regions
  - Generalized upper bounds
  - Path constraints

- Polymatroid inequalities can be improved for other feasible regions
  - Cardinality constraint
  - Knapsack constraint

Future work

Other mixed-binary structures $x \in \{0,1\}^n, \ y \in \mathbb{R}^m$

- $\min -\mu' y + \sqrt{y'\Sigma y}$ s.t $\sum_{i=1}^{n} y_i = 1, \ \sum_{i=1}^{n} x_i \leq k, \ 0 \leq y_i \leq x_i$
  - Fixed charge structure
  - **Application:** Portfolio optimization

- $\min f(y)$ s.t $-Mx_i \leq y_i \leq Mx_i, \ \sum_{i=1}^{n} x_i \leq k$
  - Sparsity
  - **Application:** Statistics (e.g. best subset regression)

Thank you!