Coalitional Stability in Matching Problems with Externalities and Random Preferences

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Classical Matching Problems

Gale and Shapley (1962) introduced the *marriage market* and the *roommate problem*, two classical models where agents may form pairs among them.

In a marriage market the population is divided in two disjoint sets. Each member on one side has preferences for the individuals on the opposite side. In this context, a matching is a distribution of individuals into couples formed by agents of different sides.

The roommate problem extends the classical marriage market framework allowing individuals to form couples without dividing the population into two disjoint sets.

Classical Matching Problems

Shapley and Scarf (1974) introduced the *housing market*, a model in which individuals may want to redistribute their houses without making transfers.

In a housing market there is a finite number of agents, each of them owning a house and having preferences for the properties in the market. In this context, a matching is a redistribution of houses among agents.

There are also matching problems that capture the *allocation* of indivisible goods among agents, as the house allocation problems of Hylland and Zeckhauser (1979), Svensson (1994), or Abdulkaridoglu and Sonmez (1999).

Matching with Externalities

These classical matching problems ignore situations where agents recognize that their wellbeing is affected by the other couples formed:

- Students internalizing peer effects on preferences to choose among colleges.
- Firms considering the structure of an industry before deciding to integrate.
- Housing markets where agents take into account the identity of their neighbors.

A simple way to include externalities in matching problems is to assume that individual preferences are defined over the set of admissible matchings.

Matching with Externalities

In the presence of externalities, it is natural to assume that coalitions evaluating a deviation consider possible reactions.

A group of agents will want to block a matching only when their welfare is improved in <u>any scenario</u> compatible with their beliefs about potential reactions.

These beliefs may depend on the matching from which deviation occurs and on the agreement that members of the coalition implement to deviate.

For instance, an extremely prudent coalition considers all the possible reactions of other individuals before implementing a deviation.

Matching with Externalities

- The presence of externalities may compromise the non-emptiness of the core in *marriage markets* and *roommate problems*.
  Sasaki and Toda (1986, 1996), Mumcu and Saglam (2010)
  Contreras and Torres-Martínez (2021)

- The presence of externalities may compromise the non-emptiness of the core in *Shapley-Scarf housing markets*.
  Mumcu and Saglam (2007), Graciano, Meo and Yannelis (2020)
  Hong and Park (2021)

- When the population grows, the negative effects of externalities in pairwise stability can be strongly mitigated.
  Braitt and Torres-Martínez (2021)

In this talk...

Without committing to a specific structure of externalities, we address coalitional stability in matching problems with externalities and random preferences.

- We show that five factors play a relevant role in the probability of solvability:
  - the *number of feasible coalitions*.
  - the *number of agreements* that coalitions' members can implement.
  - the *incidence of externalities* in preferences.
  - the *prudence of coalitions* to evaluate a deviation.
  - the *social connectedness* of those that can react to a deviation.
- We determine sufficient conditions for asymptotic coalitional stability.

Matching Problems with Externalities

A *matching problem with externalities*

$$\Gamma = [\mathcal{H}, \mathcal{P}, (\succeq^h)_{h\in\mathcal{H}}, \mathcal{H}_e, \mathcal{C}, (\mathcal{M}(S), \Theta^S)_{S\in\mathcal{C}}]$$

is characterized by:

- $\mathcal{H}$: finite set of agents that can form pairs among them.
- $\mathcal{P}$: set of admissible pairs.
- $\succeq^h$: complete and transitive preference relation of agent h, which is defined over the non-empty set of admissible matchings

$$\mathcal{M} = \{\mu : \mathcal{H} \to \mathcal{H} : (h, \mu(h)) \in \mathcal{P} \wedge \mu(\mu(h)) = h, \forall h \in \mathcal{H}\}.$$

- $\mathcal{H}_e$: non-empty set of agents whose preferences are affected by the distribution of the population in pairs.

Matching Problems with Externalities

A *matching problem with externalities*

$$\Gamma = [\mathcal{H}, \mathcal{P}, (\succeq^h)_{h \in \mathcal{H}}, \mathcal{H}_e, \mathcal{C}, (\mathcal{M}(S), \Theta^S)_{S \in \mathcal{C}}]$$

is characterized by:

- $\mathcal{C}$: set of feasible coalitions.
- $\mathcal{M}(S)$: set of admissible agreements among the members of S.
- $\Theta^{S,f}(\mu)$: set of admissible matchings that agents in $S \in \mathcal{C}$ believe that can be set up when other agents react, after they deviate from $\mu \in \mathcal{M}$ to form pairs among them as in $f \in \mathcal{M}(S)$.

Matching Problems with Externalities

An admissible matching μ is blocked by a feasible coalition S when there is an agreement $f \in \mathcal{M}(S)$ among its members such that:

(i) $\mu(h) \neq f(h)$ for some $h \in S$.

(ii) $\eta \succeq^h \mu$ for all $\eta \in \Theta^{S,f}(\mu)$ and $h \in S$.

(iii) For some $h \in S$ there exists $\nu \in \Theta^{S,f}(\mu)$ such that $\nu \succ^h \mu$.

The set of coalitionally stable matchings, denoted by $\mathbb{K}(\Gamma)$, is the collection of admissible matchings that cannot be blocked by any feasible coalition.

Matching Problems with Externalities

Roommate problems
- There is no restriction over admissible pairs: $\mathcal{P} = \mathcal{H} \times \mathcal{H}$.

Marriage markets
- $\mathcal{H} = A \cup B$, where A and B are non-empty and disjoint sets.
- $\mathcal{P} = (A \times B) \cup (B \times A) \cup \{(h, h) : h \in \mathcal{H}\}$.

Gale-Shapley marriage markets
- $\mathcal{H} = A \cup B$, where A and B are disjoint sets with $|A| = |B|$.
- $\mathcal{P} = (A \times B) \cup (B \times A)$.
- Any feasible coalition S satisfies $|S \cap A| = |S \cap B|$.

Matching Problems with Externalities

College admission problems

- $\mathcal{H} = (A_1 \cup \cdots \cup A_r) \cup B$, where A_k is a set of representatives of college $k \in \{1, \ldots, r\}$ and B is a set of students.
- Each representative of a college has a vacancy and can fill it by forming a pair with a student.
- Each student can be admitted to a college by pairing with one of its representatives or she can decide to remain outside the university system.
- Two representatives of a same college have the same preferences.
- Agents are indifferent between two matchings associating the same students' set to each college.
- Beliefs about reactions to deviations only depend on the distribution of students in colleges. That is, given $S, S' \in \mathcal{C}$ and $(f, f') \in \mathcal{M}(S) \times \mathcal{M}(S')$,

$$\left.\begin{array}{l} \text{For all } k \in \{1, \ldots, r\}: \\ \quad \#(S \cap A_k) = \#(S' \cap A_k) \\ \quad f(S \cap A_k) \cap B = f'(S' \cap A_k) \cap B \end{array}\right\} \implies \Theta^{S,f} = \Theta^{S',f'}.$$

Matching Problems with Externalities

Shapley-Scarf housing markets

Suppose that $\mathcal{H} = A \cup B$, where A is a set of individuals and B is a set houses.

Then Γ is a Shapley-Scarf housing market as long as the following properties hold:

- Each individual $a \in A$ is endowed with a house and $|A| = |B|$.
- Each house $b \in B$ has trivial preferences: $I(b, \mu) = \mathcal{M}$ for all $\mu \in \mathcal{M}$.
- The set of admissible pairs satisfies $\mathcal{P} = (A \times B) \cup (B \times A)$.
- If $S \in \mathcal{C}$, then the houses $S \cap B$ are the endowments of the individuals $S \cap A$.

Determinants of Coalitional Stability

Some notation:

$I(h,\eta)$	:	Indifference class of $\succeq^h$ containing η.
$I(h,\mathcal{M}')$	:	Indifference classes of $\succeq^h$ intersecting $\mathcal{M}' \subseteq \mathcal{M}$.
$\mathcal{M}^{S,f}$	:	Matchings in which members of S are distributed as in $f \in \mathcal{M}(S)$.

Note that,

$$|I(h,\Theta^{S,f}(\mu))| = \frac{|I(h,\Theta^{S,f}(\mu))|}{|I(h,\mathcal{M}^{S,f})|} \frac{|I(h,\mathcal{M}^{S,f})|}{|\mathcal{M}^{S,f}|} |\mathcal{M}^{S,f}|$$

Determinants of Coalitional Stability

Consider the following measures:

- An index $\pi_{pr}(\Gamma) \in (0,1]$ that quantifies the *prudence of coalitions*:

$$\pi_{pr}(\Gamma) \quad = \quad \min_{\mu \in \mathcal{M}} \min_{S \in \mathcal{C}} \min_{f \in \mathcal{M}(S)} \min_{h \in S \cap \mathcal{H}_e} \frac{|I(h, \Theta^{S,f}(\mu))|}{|I(h, \mathcal{M}^{S,f})|}.$$

- An index $\pi_{ex}(\Gamma) \in (0,1]$ that quantifies the *incidence of externalities in preferences*:

$$\pi_{ex}(\Gamma) \quad = \quad \min_{S \in \mathcal{C}} \min_{f \in \mathcal{M}(S)} \min_{h \in S \cap \mathcal{H}_e} \frac{|I(h, \mathcal{M}^{S,f})|}{|\mathcal{M}^{S,f}|}.$$

- An index $\pi_{sc}(\Gamma) \in [1,+\infty)$ that quantifies the *social connectedness of agents that can react to a deviation*:

$$\pi_{sc}(\Gamma) \quad = \quad \min_{S \in \mathcal{C}} \min_{f \in \mathcal{M}(S)} |\mathcal{M}^{S,f}|.$$

Random Preferences

For each agent h, the indifference classes $\{I(h,\mu)\}_{\mu\in\mathcal{M}}$ are exogenously given.

However, the way in which these indifference classes are ordered is randomly determined.

Given $\mu,\eta\in\mathcal{M}$ such that $\mu\notin I(h,\eta)$,

$$\mu \succ^h \eta \qquad \Longleftrightarrow \qquad X_h^{I(h,\mu)} > X_h^{I(h,\eta)}.$$

The random variables $\{X_h^{I(h,\mu)}\}_{(h,\mu)\in\mathcal{H}\times\mathcal{M}}$ are independent.
Each $X_h^{I(h,\mu)}$ has a continuous probability distribution $G_h^{I(h,\mu)}:\mathbb{R}\to[0,1]$.

Random Preferences

Let $\Omega^h(\Gamma)$ be the set of matchings $\mu \in \mathcal{M}$ such that $X_h^{I(h,\mu)}$ *first-order stochastically dominates* the random variables in the set

$$\bigcup_{S \in \mathcal{C}:\, h \in S} \left\{ X_h^{I(h,\eta)} : \eta \in \Theta^{S,f}(\mu),\, f \in \mathcal{M}(S) \right\}.$$

Assumption. $\bigcap_{h \in \mathcal{H}} \Omega^h(\Gamma)$ is non-empty.

Examples

(1) For each agent h, $\{X_h^{I(h,\mu)}\}_{\mu \in \mathcal{M}}$ are identically distributed random variables.

(2) Each agent h has a set $\mathcal{H}^h \subseteq \{k \in \mathcal{H} : (k,h) \in \mathcal{P}\}$ of *weakly preferred potential partners* in the sense that $\{X_h^{I(h,\mu)}\}_{\mu(h) \in \mathcal{H}^h}$ are identically distributed and first-order stochastically dominate the random variables $\{X_\eta^{I(h,\eta)}\}_{\eta(h) \notin \mathcal{H}^h}$.

In this context, $\{\mu \in \mathcal{M} : \mu(h) \in \mathcal{H}^h\} \subseteq \Omega^h(\Gamma)$ for all $h \in \mathcal{H}$. Therefore, the assumption above holds when there is an admissible matching that pairs each agent with one of her weakly preferred partners.

Lower Bounds for the Probability of Solvability

Theorem 1. If Γ is a matching problem where each feasible coalition has at least one member with strict preferences, then

$$\mathbb{P}[\mathbb{K}(\Gamma)] \quad \geq \quad 1 - \frac{1}{\pi(\Gamma)} \sum_{S \in \mathcal{C}} |\mathcal{M}(S)|.$$

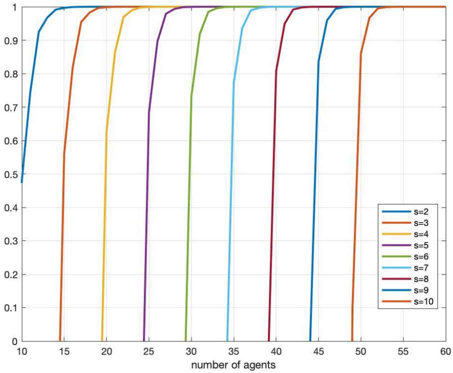
Example. Suppose that Γ is a *roommate problem* in which preferences are strict and coalitions are extremely prudent. Then, $\pi_{pr}(\Gamma) = \pi_{ex}(\Gamma) = 1$.

Moreover, if $\bar{s}$ is the maximal size of feasible coalitions, then $\pi_{sc}(\Gamma) \geq (|\mathcal{H}| - \bar{s})!!$.

Therefore, it follows from Theorem 1 that

$$\mathbb{P}[\mathbb{K}(\Gamma)] \quad \geq \quad 1 - \frac{|\mathcal{H}|^{\bar{s}}\, \bar{s}!}{(|\mathcal{H}| - \bar{s})!!}.$$

Lower Bounds for the Probability of Solvability



Lower bounds for $\mathbb{P}[\mathbb{K}]$ *in Roommate Problems with strict preferences when feasible coalitions are extremely prudent and have at most* s *members.*

Asymptotic Coalitional Stability

Let $\mathbb{T}$ be the collection of sequences $\{\Gamma_n\}_{n\in\mathbb{N}}$ of matching problems satisfying the following properties:

- The sequence $\{\pi_{pr}(\Gamma_n)\}_{n\in\mathbb{N}}$ decays at most exponentially.
- Each feasible coalitions in Γ_n has a member with strict preferences and the sequence $\{\pi_{ex}(\Gamma_n)\}_{n\in\mathbb{N}}$ decays at most exponentially.
- The number of agents in Γ_n, $|\mathcal{H}_n|$, goes to infinity as n increases.

Asymptotic Coalitional Stability

Theorem 2. Let $\{\Gamma_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ be a sequence of matching problems such that, for some $\sigma, \delta \in (0,1)$, the following conditions hold for each n large enough:

(a) Feasible coalitions in Γ_n have at most $\sigma|\mathcal{H}_n|$ members.

(b) Every agent in Γ_n has at least $\delta|\mathcal{H}_n|$ admissible partners.

Then $\{\Gamma_n\}_{n\in\mathbb{N}}$ is asymptotically coalitionally stable as long as $\sigma < \delta/2$.

Theorem 3. Let $\{\Gamma_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ be a sequence of matching problems satisfying the conditions (a)-(b) of Theorem 2 for some $\sigma, \delta \in (0,1)$, with $\sigma < \delta$.

Then $\{\Gamma_n\}_{n\in\mathbb{N}}$ is asymptotically coalitionally stable as long as coalitions' agreements increase at most exponentially.

Asymptotic Coalitional Stability

Corollary [Roommate Problems]

Let $\{\Gamma_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ be a sequence of roommate problems such that, for some $\sigma \in (0,1)$ and for n large enough, feasible coalitions in Γ_n have at most $\sigma|\mathcal{H}_n|$ members.

Then $\{\Gamma_n\}_{n\in\mathbb{N}}$ is asymptotically coalitionally stable when either $\sigma < 1/2$ or coalitions' agreements increase at most exponentially.

Asymptotic Coalitional Stability

Theorem 4 [Gale-Shapley Marriage Markets]

Let $\{\Gamma_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ be a sequence of Gale-Shapley marriage markets such that, for some $\sigma \in (0,1)$ and for n large enough, feasible coalitions in Γ_n have at most $\sigma|\mathcal{H}_n|$ members. Then

$$\lim_{n\to+\infty} \mathbb{P}[\mathbb{K}(\Gamma_n)] = 1$$

as long as $\sigma < 1/4$ or coalitions' agreements increase at most exponentially.

Theorem 5 [Shapley-Scarf Housing Markets]

Let $\{\Gamma_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ be a sequence of Shapley-Scarf housing markets where each Γ_n has n individuals and n houses. Suppose that for some $\rho \in (0,1)$ and for n large enough, feasible coalitions in Γ_n include at most ρn individuals.

Then,

$$\lim_{n\to+\infty} \mathbb{P}[\mathbb{K}(\Gamma_n)] = 1$$

as long as $\rho < 1/4$ or coalitions' agreements increase at most exponentially.

Summary

- We found sufficient conditions for asymptotic coalitional stability in matching problems with externalities.
- In our main results, it was assumed that agents have a limited capacity to coordinate to form *large* coalitions.
- Asymptotic stability holds when the size of a feasible coalition never exceeds one half of the minimal number of admissible partners.
- Asymptotic stability also holds when coalitions' sizes never exceed the minimal number of admissible partners and the number of coalitions' agreements increase at most exponentially.
- The two-sided structure of *Gale-Shapley marriage markets* and *Shapley-Scarf housing markets* guaranteed that asymptotic stability holds even when coalitions have more members than agents have admissible partners.