Epidemics, production and saving. Why saving is important? (artículo en preparación)

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Why are there countries richer than others?

- Erik S. Reinert shows how rich countries developed through a combination of government intervention, protectionism, and strategic investment—rather than through free trade. ("How Rich Countries Got Rich and Why Poor Countries Stay Poor" [Rei07]).
- There are many theories about extreme poverty: as a personal failing, as a structural failing, as cultural characteristics, as a restriction of opportunities, ...
- We identify with theory about mechanisms through which the environment can influence the dynamics of poverty via disease feedbacks.
  [BKRS09, NPM+14, GSR+17]

Extreme poverty vs. Neglected Tropical Diseases NTDs

- Extreme poverty is defined as living with per capita household below 1.90 international dollars per day (in 2011).
- Neglected tropical diseases (NTDs): a set of communicable diseases (parasitic, viral and bacterial infections) that prevail in tropical and subtropical conditions in 149 countries, affect more than one billion people.

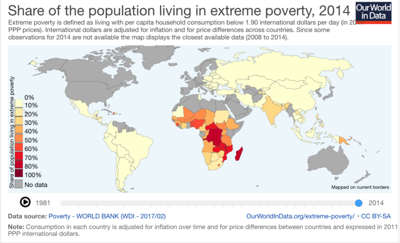

Figure: Extreme poverty

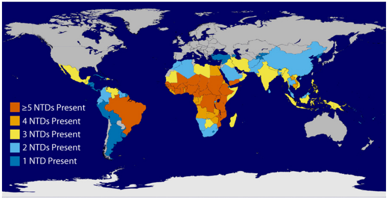

Figure: Global Overlap of the six of the common NTDs.

Specifically: Guinea worm disease, Lymphatic filariasis, Onchocerciasis, Schistosomiasis, Soil-transmitted helminthiases and Trachoma.
Source: World Bank and Centers for Disease Control and Prevention (CDC).
http://data.worldbank.org
https://www.cdc.gov/globalhealth/ntd/diseases/ntd-worldmap-static.html

Neglected Tropical Diseases NTDs (WHO)

- Buruli ulcer
- Chagas disease ⋆
- Dengue and Chikungunya
- Dracunculiasis (guinea-worm disease)
- Echinococcosis
- Foodborne trematodiases
- Human African trypanosomiasis (sleeping sickness)
- Leishmaniasis
- Leprosy (Hansen's disease)
- Lymphatic filariasis
- Mycetoma, chromoblastomycosis and other deep mycoses
- Onchocerciasis (river blindness)
- Rabies
- Scabies and other ectoparasites
- Schistosomiasis
- Soil-transmitted helminthiases
- Snakebite envenoming ⋆
- Taeniasis/Cysticercosis
- Trachoma
- Yaws (Endemic treponematoses)

Question

Is there an optimal level of savings that maximizes the output product of an economy under the presence of recurrent diseases?

Answer

Yes!

To obtain this answer, we use a mathematical model that results from the coupling of an epidemiological model and an economic model.

The epidemiological model is a SIS type model and the economic model is a closed economic Solow [Sol56] type model with a Cobb-Douglas [CD28] production function.

Assumptions

- We will not study a specific disease but a set of diseases with the following characteristics:
  - affect the same population and are recurrent or emerging,
  - are temporary or of short duration,
  - are infectious caused by biological agents, such as bacteria, viruses, fungi, etc.
  - or are transmitted by contagion from one individual to another or by the ingestion of contaminated food or water. Transited by vectors: Aedes (Dengue fever, Rift Valley fever, Yellow fever, Chikungunya); Anopheles (Malaria); Culex (Japanese encephalitis, Lymphatic filariasis, West Nile fever); Sandflies (Leishmaniasis, Sandfly fever); etc.
  - not all are acquired immunity.
- The set of diseases could be thought of: malaria, cholera, dengue, chicungunya, chagas, leishmaniasis, lymphatic filiariasis, respiratory diseases, sinusitis, pneumonia, meningitis, syphilis, tuberculosis, tetanus, and diarrhea. [WHO07]
- We will use a Kermack-McKendrick SIS model to model the population affected by the disease.

Further assumptions

- The economy is closed, i.e. with no international trade and without government expenditure.
- We will also suppose that the product of the economy ($\tilde{Y} = \tilde{Y}(t)$) is given by the influence of the input capital ($\tilde{K} = \tilde{K}(t)$) and by the susceptible population ($\tilde{S} = \tilde{S}(t)$).
- We will use the Solow-Swan economic growth model with Cobb-Douglas aggregate production function to model the economy of the population affected by the disease: $\tilde{Y} = c\tilde{K}^{\epsilon}\tilde{S}^{1-\epsilon}$
- The change in capital stock is assumed as $\tilde{K}' = a\tilde{Y} - \delta\tilde{K}$, i.e. the stock of capital depreciates over time at a constant rate δ while only a fraction of the output ($\tilde{M} = (1-a)\tilde{Y}(t)$) with $0 < a < 1$ is consumed, leaving a saved share a for investment that we will assume the portion saved ($a\tilde{Y}$).
- We finally assume that the portion saved ($a\tilde{Y}$) is invested back to capital.

$\tilde{S}$ = Susceptible population

$\tilde{S}$ = Susceptible population
$\tilde{I}$ = Infected population

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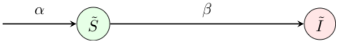
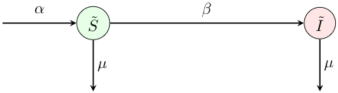
$\tilde{I}$ = Infected population
α = Natural rate of birth



$\tilde{S}$ = Susceptible population
$\tilde{I}$ = Infected population
α = Natural rate of birth
β = Transmission rate



$\tilde{S}$ = Susceptible population
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α = Natural rate of birth
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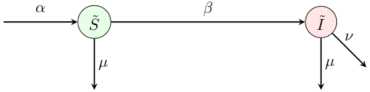
μ = Natural rate of death



$\tilde{S}$ = Susceptible population
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α = Natural rate of birth
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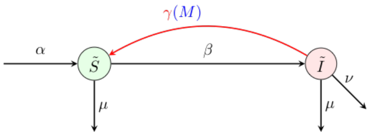

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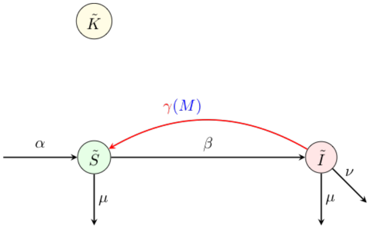
ν = Death rate cased by disease
γ = Disease recovery rate
M = Income intended for consumption



$\tilde{K}$ = Input capital into the economy

$\tilde{S}$ = Susceptible population
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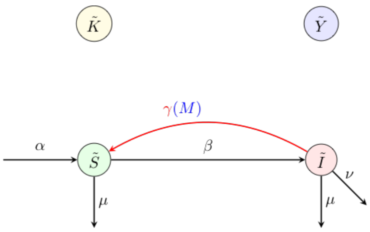

$\tilde{K}$ = Input capital into the economy (\$)
$\tilde{Y}$ = Product of the economy (\$)

$\tilde{S}$ = Susceptible population
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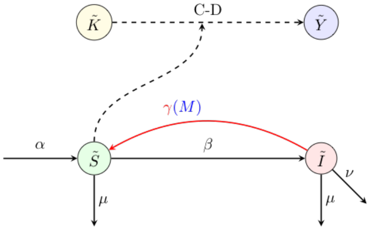

$\tilde{K}$ = Input capital into the economy ($)
$\tilde{Y}$ = Product of the economy ($)
C-D = Cobb-Douglas prod. function

$\tilde{S}$ = Susceptible population
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α = Natural rate of birth
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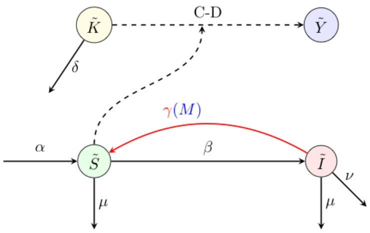

$\tilde{K}$ = Input capital into the economy ($)
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C-D = Cobb-Douglas prod. function
δ = Capital depreciation

$\tilde{S}$ = Susceptible population
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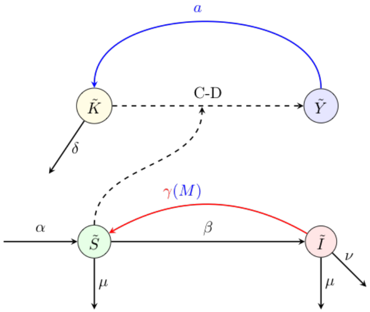

$\tilde{K}$ = Input capital into the economy ($)
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C-D = Cobb-Douglas prod. function
δ = Capital depreciation
a = Portion intended for savings

$\tilde{S}$ = Susceptible population
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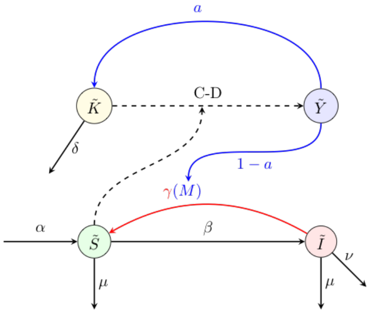

$\tilde{K}$ = Input capital into the economy ($)
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δ = Capital depreciation
a = portion intended for savings
$1 - a$ = portion intended for consumption

$\tilde{S}$ = Susceptible population
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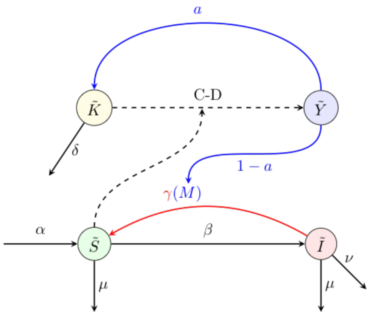


$\tilde{K}$ = Input capital into the economy ($)

$\tilde{Y}$ = Product of the economy ($)

C-D = Cobb-Douglas $\boxed{\tilde{Y} = c\tilde{K}^{\epsilon}\tilde{S}^{1-\epsilon}}$

δ = Capital depreciation

a = portion intended for savings

$1 - a$ = portion intended for consumption

$\tilde{S}$ = Susceptible population

$\tilde{I}$ = Infected population

α = Natural rate of birth

β = Transmission rate

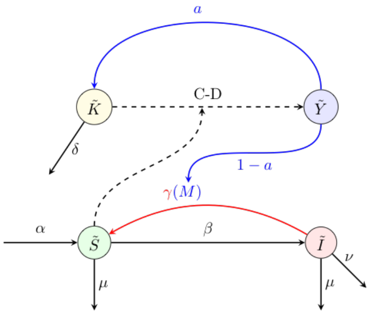
μ = Natural rate of death

ν = Death rate caused by disease

γ = Disease recovery rate

M = Income intended for consumption

$$\begin{cases} \tilde{S}' = \alpha\tilde{N} + \gamma(M)\tilde{I} - \left(\beta\dfrac{\tilde{I}}{\tilde{N}} + \mu\right)\tilde{S} \\ \tilde{I}' = \beta\tilde{S}\dfrac{\tilde{I}}{\tilde{N}} - (\mu + \nu + \gamma(M))\,\tilde{I} \\ \tilde{K}' = a\tilde{Y} - \delta\tilde{K} \\ \tilde{Y}' = c\,(1-\epsilon)\,\tilde{K}^{\epsilon}\tilde{S}^{-\epsilon}\tilde{S}' + c\epsilon\tilde{S}^{1-\epsilon}\tilde{K}^{\epsilon-1}\tilde{K}' \end{cases}$$



$\tilde{K}$ = Input capital into the economy ($)

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C-D = Cobb-Douglas $\boxed{\tilde{Y} = c\tilde{K}^{\epsilon}\tilde{S}^{1-\epsilon}}$

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a = portion intended for savings

$1-a$ = portion intended for consumption

$\tilde{S}$ = Susceptible population

$\tilde{I}$ = Infected population

α = Natural rate of birth

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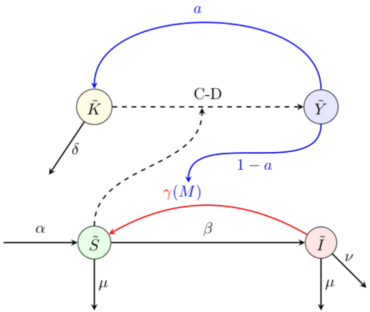
M = Income intended for consumption

$$S = \tilde{S}/\tilde{N};\ I = \tilde{I}/\tilde{N}; \quad K = \tilde{K}/\tilde{N};\ Y = \tilde{Y}/\tilde{N}; \quad \boxed{\gamma(M) = \frac{\tau M\bar{h} + \kappa\bar{\gamma}}{M+\kappa};} \quad \boxed{M = (1-a)Y}$$

$$\begin{cases} \tilde{S}' = \alpha\tilde{N} + \gamma(M)\tilde{I} - \left(\beta\dfrac{\tilde{I}}{\tilde{N}} + \mu\right)\tilde{S} \\ \tilde{I}' = \beta\tilde{S}\dfrac{\tilde{I}}{\tilde{N}} - (\mu + \nu + \gamma(M))\,\tilde{I} \\ \tilde{K}' = a\tilde{Y} - \delta\tilde{K} \\ \tilde{Y}' = c\,(1-\epsilon)\,\tilde{K}^{\epsilon}\tilde{S}^{-\epsilon}\tilde{S}' + c\epsilon\tilde{S}^{1-\epsilon}\tilde{K}^{\epsilon-1}\tilde{K}' \end{cases}$$



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$$S = \tilde S/\tilde N;\ I = \tilde I/\tilde N; \quad K = \tilde K/\tilde N;\ Y = \tilde Y/\tilde N; \quad \boxed{\gamma(M) = \frac{\tau M\bar h + \kappa\bar\gamma}{M+\kappa};} \quad \boxed{M = (1-a)Y}$$

$$\begin{cases} \tilde S\,' = \alpha\tilde N + \gamma(M)\tilde I - \left(\beta\dfrac{\tilde I}{\tilde N} + \mu\right)\tilde S \\ \tilde I\,' = \beta\tilde S\dfrac{\tilde I}{\tilde N} - (\mu+\nu+\gamma(M))\,\tilde I \\ \tilde K\,' = a\tilde Y - \delta\tilde K \\ \tilde Y\,' = c\,(1-\epsilon)\,\tilde K^{\epsilon}\tilde S^{-\epsilon}\tilde S\,' + c\epsilon\tilde S^{1-\epsilon}\tilde K^{\epsilon-1}\tilde K\,' \end{cases} \Rightarrow \begin{cases} S\,' = \alpha - \alpha S - \beta SI + \gamma(M)I + \nu IS \\ I\,' = \beta SI - (\alpha+\nu+\gamma(M))\,I + \nu I^2 \\ K\,' = a\,Y - (\delta+\alpha-\mu-\nu I)\,K \\ Y\,' = c\,(1-\epsilon)\,K^{\epsilon}S^{-\epsilon}S\,' + c\epsilon S^{1-\epsilon}K^{\epsilon-1}K\,' \end{cases}$$

Equilibria and $\mathcal{R}_0$

"Disease free equilibrium (DFE)", $(S_1^*, I_1^*, K_1^*, Y_1^*)$:

$$S_1^* = 1, \quad I_1^* = 0, \quad K_1^* = \left(\frac{a\,c}{\delta + \alpha - \mu}\right)^{1/(1-\epsilon)}, \quad Y_1^* = c^{1/(1-\epsilon)} \left(\frac{a}{\delta + \alpha - \mu}\right)^{\epsilon/(1-\epsilon)}$$

Formulas relating "endemic equilibrium", $(S_2^*, I_2^*, K_2^*, Y_2^*)$:

$$\begin{cases} S_2^* = \dfrac{\alpha + \gamma^*(M)}{\beta - \nu} \\ I_2^* = 1 - \dfrac{\alpha + \gamma^*(M)}{\beta - \nu} \\ K_2^* = S_2^* \left(\dfrac{a\,c}{\delta + \alpha - \mu - \nu(1 - S_2^*)}\right)^{1/(1-\epsilon)} \\ Y_2^* = c^{1/(1-\epsilon)} S_2^* \left(\dfrac{a}{\delta + \alpha - \mu - \nu\,(1 - S_2^*)}\right)^{\epsilon/(1-\epsilon)} \end{cases}$$

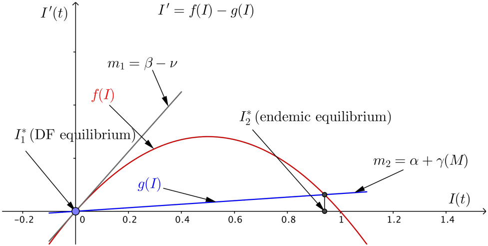
$$\mathcal{R}_0 = \frac{\beta}{\alpha + \nu + \gamma^*(M)}$$

Stability of I^*

Theorem (Stability of I^*)

- *If $\mathcal{R}_0 < 1 \Rightarrow \beta < \alpha + \nu + \overline{\gamma}$, then I_1^* es asymptotically stable.*
- *If $\mathcal{R}_0 > 1 \Rightarrow \beta > \alpha + \nu + \tau\overline{h}$, then I_2^* es asymptotically stable.*

Proof.



$$\overline{\gamma} < \gamma(M) < \tau\overline{h}$$

$$I' = \beta S I - (\alpha + \nu + \gamma(M))\, I + \nu I^2 = 0$$

$$I' = (\beta - \nu) I (1 - I) - (\alpha + \gamma(M))\, I = f(I) - g(I) = 0$$

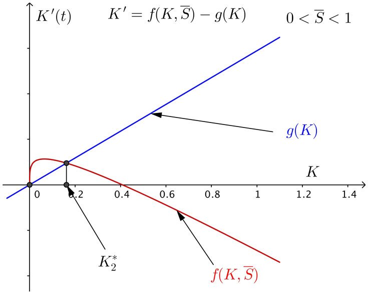
□

Stability of K_2^*

Theorem (Stability of K_2^*)

- *If $\alpha + \delta > \mu + \nu$, then K_2^* is asymptotically stable.*

Proof.



$$K' = \left[acK^{\epsilon} S^{1/(1-\epsilon)} - \nu KS\right] - \left[(\delta + \alpha - \mu - \nu) K\right] = f(K, S) - g(K) = 0$$

- $\lim_{K \to 0} \dfrac{\partial f}{\partial K} = +\infty \qquad \dfrac{\partial^2 f}{\partial K^2} < 0 \qquad x\text{-intercept } K = 0 \text{ and } K = \left(\dfrac{ac}{\nu \overline{S}^{\epsilon}}\right)^{1/(1-\epsilon)}$

□

La función $Y^* = Y^*(a)$

$$\begin{cases} \gamma^*(a) = \dfrac{\tau \bar{h}(1-a)Y^* + \kappa\bar{\gamma}}{(1-a)Y^* + \kappa}; \\ S^* = \dfrac{\alpha + \gamma^*(a)}{\beta - \nu} \\ Y^* = c^{1/(1-\epsilon)} S^* \left(\dfrac{a}{\delta + \alpha - \mu - \nu(1-S^*)} \right)^{\epsilon/(1-\epsilon)} \end{cases}$$

Theorem 3

Sea $\nu = 0$ y $\alpha = \mu$. Definimos Q_0 por:

$$\boxed{Q_0 = \left[\left(\frac{1-\epsilon}{\epsilon} \right) \left(c^{1/(1-\epsilon)} \right) \left(\frac{1}{\delta} \right)^{\epsilon/(1-\epsilon)} \right] \left[\frac{\tau \bar{h} - \bar{\gamma}}{\beta\kappa} \right]}$$

La función $Y^* = Y^*(a)$ es continua en $[0, 1]$ y diferenciable en $(0, 1)$ y satiface:

$$\lim_{a \to 0^+} \frac{dY^*}{da} = +\infty \qquad \lim_{a \to 1^-} \frac{dY^*}{da} = L$$

Si

$$\mathcal{R}_0 > 1 \quad \text{y} \quad Q_0 > 1 \Rightarrow L < 0$$

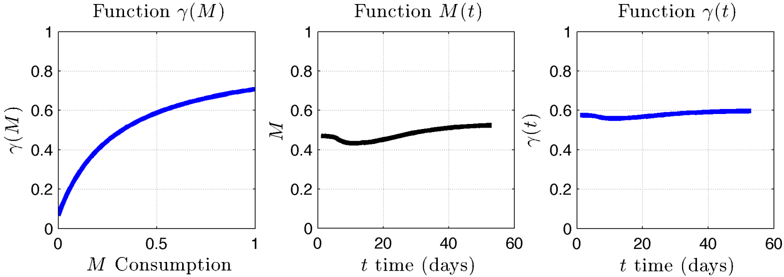
En este caso hay una recuperación de la economía y por lo tanto $Y^* = Y^*(a)$ alcanza un máximo para algún valor de $a \in (0, 1)$

Example: Simulation for Nigeria (Hypothetical case)

Table:

Parameter	Interpretation	Value
α	Natural rate of birth	4.7318×10^{-5}
β	Transmission rate	1.1250
μ	Natural rate of death	5.2576×10^{-5}
ν	Additional death rate caused by disease	5.8156×10^{-5}
$\overline{h}$	Maximum level of nutrition attainable	0.9
$\overline{\gamma}$	Rate of natural recovery without any intervention	0.0667
κ	Half saturation constant	0.30
c	total factor productivity	0.8352
a	Proportion of product saved	0.1
$1 - a$	Proportion of product intended for consumption	0.9
ϵ	Elasticity of substitution of capital in the product	0.25
δ	Average depreciation rate of the capital stock	0.0590
τ	Exogenous parameter (calibration)	1.2
S_0	Initial susceptible population in proportion	0.8
I_0	Initial infectious population	0.2
K_0	Initial susceptible population in proportion	0.3
Y_0	Initial infectious population	0.5229

Recovery rate as a function of M, $\gamma(M)$



Sensitivity indices of $\mathcal{R}_0$

Definition (Normalized forward sensitivity index)

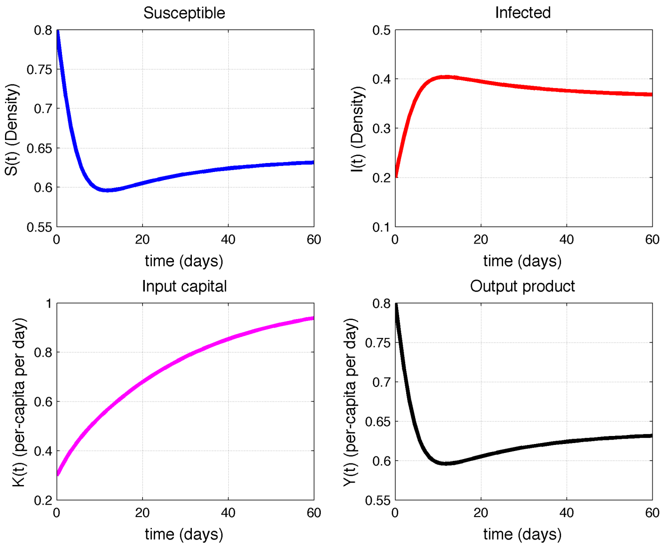
The normalized forward sensitivity index of $\mathcal{R}_0$, that depends differentiably on a parameter p, is defined by

$$\Upsilon_p^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0}$$

Table: Sensitivity indices of $\mathcal{R}_0$ evaluated at the baseline parameter values given in Table 1

Parameter	**Sensitivity index**
α	-0.0007085
β	$+1$
γ	-0.998317
ν	-0.000974275

System solutions



Steady state of production as a function of saving level $Y^* = Y^*(a)$

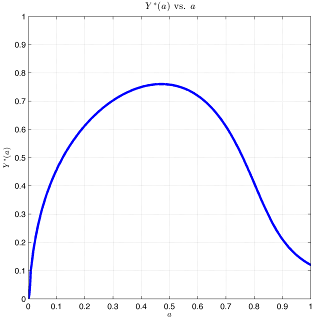


Figure: For a level of savings of approximately 50%, the maximum output of the economy is obtained.

Conclusiones

- Se supone inicialmente una población con una economía en equilibrio y la cual no está afectada por alguna enfermedad.
- Al aparecer un brote epidémico de una enferemedad, se quiere contestar la pregunta: ¿es posible una recuperación de la economía cuando ésta es afectada por la epidemia?
- Al analizar cómo una epidemia puede influir en la economía de una comnidad se establece el acoplamiento de dos modelos matemáticos y se obtiene un sistema dinámico autónomo. La componente epidemiológica se modela con un sistema SIS y la componente económica mediante un modelo de Solow-Swan con función de producción tipo Cobb-Douglas.
- El acoplamiento está dado de dos maneras: se asume que la población activa en el proceso de producción es la población de susceptibles y mediante la porción del producto destinado al consumo. Se asume que esta porción influye de manera directa en la recuperación de la enfermedad por mejoramiento en su nutrición.
- Mediante unos supuestos básicos que deben cumplir los parámetros se establece la existencia de un punto de equilibrio endémico asintóticamente estable.
- Relajando algunas condiciones, más precisamente: la epidemia ocurre en un período de tiempo corto (máximo 2 meses) y que en ese período la población permanece constante, es decir que las tasas de nacimiento y de muerte son iguales, se demuestra que el equilibrio del producto es un función del nivel de ahorro la cual alcanza un máximo bajo ciertas restricciones de los parámetros.

Conclusiones (cont.)

- El parámetro de contagio es el parámetro decisorio a la hora de la recuperación de la economía. Por ejemplo si este parámetro es tal que no hay epidemia entonces, no hay afectación en la producción.

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Thank you!